Orthogonality Check and Correction of Measured Modes

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The self-compatibility of a set of measured vibration modes usually is checked by resort to the orthogonality characteristics of true normal modes. This paper addresses itself to the information content in the orthogonality check and shows that, to first order, only symmetric error distributions in modal matrices are revealed by the check. The significance of the inability to determine antisymmetric errors is discussed. The influence of the truncation of the measured modal set upon the information content of the truncated orthogonality check is reviewed. Methods for eliminating symmetric error distributions are developed and the results of some artificially generated test cases are presented.

Nomenclature

C	= corruption matrix
E	= elastic (spring) matrix associated with M and Y
I	= identity matrix
M	= mass matrix associated with E and Y
$[OR]$	= orthogonality check matrix
X	= orthonormal set of modes
X_m, Y_m, etc.	= measured values of X, Y, etc.
X^T, Y^T, etc.	= transpose of X, Y, etc.
Y	= normal modes of vibration of a structure
α	= symmetric portion of δ
β	= skew symmetric portion of δ
δ	= defined by $C = I + \delta$
λ	= matrix of modal eigenvalues associated with Y

I. Introduction

MODAL data are of great importance in the design of spacecraft as dynamic loads set the principal structural requirements. Raw measured data obtained from modal surveys of these very complex, asymmetric structures usually contain errors due to the basic difficulty of experimental mode separation. Gladwell and Bishop[1] present a critical summary of various methods of resonance testing. Methods of deriving modal properties from analysis of the response of transiently excited structures are in current vogue.[2] Nonetheless, residual errors of importance may remain in the set of "measured" modes finally made available to the analyst. It is standard practice to check the self-compatibility of this final set by resorting to the orthogonality characteristics of true normal modes. (Corollary to this practice is the tacit assumption that the mass matrix employed is accurate.)

When the orthogonality check is made, it is found that nonzero, off-diagonal elements exist. With normalization such that the main diagonal elements are all unity, one can categorize the off-diagonal elements. Values below about $|0.05|$ may be considered "small" and are usually ignored. Values above about $|0.5|$ must be considered so large that a fundamental error in the measurements must be assumed, and the modal set must be considered nonusable if specific, correctable, procedural errors cannot be identified. Usually, however, none of the off-diagonal elements are very large; most are quite small, whereas some are of moderate size.

It is when the orthogonality check reveals an intermediate state that the analyst is put in a quandary. The errors are too large to neglect but not large enough to throw substantial doubt on the general validity of the modes. In such a case, the analyst may attempt to reduce the off-diagonal terms in accordance with some schedule based on an approximate vibration theory. Some analysts, having greater faith in the measured mode data than in the mass data, make use solely of the experimental modal data to define new mass matrices such that the orthogonality check is satisfied.[3-5] Others rely on the analytical mass matrix and alter the measured modes.[6-8] Unfortunately, an infinitude of modal sets can be found, each of which will satisfy the orthogonality check perfectly.

Not addressed in the references are certain basic limitations in the information content of the orthogonality check. This paper applies itself to that subject and makes some suggestions based thereon. Primary to the arguments is the assumption that the measured modes are very nearly the correct modes of the structure that was tested and that only the minimum required changes to these measured modes should be made, consistent with the acceptance of the accuracy of the analytical mass matrix.

II. Complete Modal Set

Consider first the correct modal matrix, forming a complete set over the coordinate space. Normally, of course, the measured modes will be an incomplete set. If the matrix Y is that set properly normalized, then orthogonality of the modes results in $Y^T M Y \equiv I$, where M is the mass matrix. In general, the measured modes contain errors so that, even if all of the modes were measured, we would find that $Y_m = YC$, where C is a corruption matrix. That is, inasmuch as Y forms a complete set, any Y_m can be considered as having been synthesized by some coupling of the Y modes.

If an orthogonality check is made using the measured modes, one gets

$$[OR] \equiv Y_m^T M Y_m = C^T Y^T M Y C = \mathrm{C}^\mathrm{T} \mathrm{I} \mathrm{C} = \mathrm{C}^\mathrm{T} \mathrm{C} \qquad (1)$$

where $[OR]$ is the orthogonality matrix. (If Y_m is a normalized matrix, so is C.)

III. Small Errors

Consider first the case of very small errors, such that C can be written as $C = I + \delta$. Then,

$$[OR] = (I + \delta)^T (I + \delta)$$

$$= I + \delta + \delta^T + \mathcal{O}(\delta^2)$$

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Note that, to first order, no antisymmetric elements appear in the orthogonality check. That is, if

$$\delta \equiv \alpha + \beta$$

where $\alpha = \alpha^T$ and $\beta = -\beta^T$, then

$$\delta + \delta^T = \alpha + \beta + \alpha - \beta = 2\alpha$$

Thus, we conclude that any corrections to Y_m based on $[OR]$ can correct only symmetric errors.† If only a symmetric error is present, $[OR]$ can be used to determine α.

$$[OR] = I + \delta + \delta^T = I + 2\alpha \quad \alpha = \tfrac{1}{2}([OR] - I) \tag{2}$$

Note that, if δ^2 is indeed small, the solution for α is a unique solution.

IV. Significance of Neglect of β

One now can ask if the inability to determine the antisymmetric portion of C is of significance, assuming that β is of same degree of smallness as α. Let

$$C = I + \alpha + \beta, \qquad Y_m = Y(I + \alpha + \beta)$$

In the modal approach, we are diagonalizing a matrix: the elastic matrix E. The fundamental question, then, is how well do our modes (and frequencies) resynthesize the matrix? Consider $E \equiv MY\lambda Y^T M$. Then, $E_m \equiv MY_m \lambda_m Y_m^T M$. For present purposes, we assume $\lambda \equiv \lambda_m$. Then, to first order,

$$\begin{aligned} E_m &= MYC\lambda C^T Y^T M \\ &= MY(I + \alpha + \beta)\lambda(I + \alpha - \beta) Y^T M \\ &= MY\{\lambda + (\alpha\lambda + \lambda\alpha) + (\beta\lambda - \lambda\beta)\} Y^T M \\ &= E + E_\alpha + E_\beta \end{aligned}$$

Consider

$$E_\beta = MY(\beta\lambda - \lambda\beta) Y^T M$$

The (i,j) element of the matrix $\beta\lambda - \lambda\beta$ is

$$\beta_{ij}\lambda_{jj} - \lambda_{ii}\beta_{ij} = \beta_{ij}(\lambda_{jj} - \lambda_{ii})$$

If we can assume that the primary errors in the measured modes are due to inability to separate modes relatively close in frequency, then it follows that $(\lambda_{jj} - \lambda_{ii})$ is small whenever β_{ij} is not very small. Thus, the product $\beta_{ij}(\lambda_{jj} - \lambda_{ii})$ tends to be of second order, and correction of the symmetric errors [now easily seen to be $\alpha_{ij}(\lambda_{jj} + \lambda_{ii})$] takes care of all first-order errors.

V. Truncated Modal Set

Start off by partitioning a complete set of Y_m

$$[Y_{m_1} \mid Y_{m_2}] \equiv [Y_1 \mid Y_2] \begin{bmatrix} I_1 + \delta_{11} & \delta_{12} \\ \delta_{21} & I_2 + \delta_{22} \end{bmatrix}$$

$$[OR] = \begin{bmatrix} I_1 + \delta_{11}^T & \delta_{21}^T \\ \delta_{12}^T & I_2 + \delta_{22}^T \end{bmatrix} \begin{bmatrix} Y_1^T \\ Y_2^T \end{bmatrix}$$

$$[M] \; [Y_1 \mid Y_2] \begin{bmatrix} I_1 + \delta_{11} & \delta_{12} \\ \delta_{21} & I_2 + \delta_{22} \end{bmatrix}$$

†Note that an orthonormal set represents the direction cosines of one set of rectangular axes with respect to a fixed set of rectangular axes in N space. Postmultiplication by $I + \beta$ (with β small and skew-symmetric) represents a legitimate rotation of the axes in N space, maintaining their orthogonality.

and, to first order,

$$[OR] = \begin{bmatrix} I + \delta_{11} + \delta_{11}^T & \delta_{12} + \delta_{21}^T \\ \delta_{12}^T + \delta_{21} & I_2 + \delta_{22} + \delta_{22}^T \end{bmatrix}$$

Now, if we had started out with only

$$Y_{m_1} \equiv [Y_{m_1} \mid Y_{m_2}] \begin{bmatrix} I_1 \\ 0 \end{bmatrix} = [Y_1 \mid Y_2] \begin{bmatrix} I_1 + \delta_{11} \\ \delta_{21} \end{bmatrix}$$

and the truncated orthogonality check becomes

$$[OR_T] \equiv Y_{m_1}^T M Y_{m_1} = [I + \delta_{11}^T \mid \delta_{21}^T] \begin{bmatrix} I + \delta_{11} \\ \delta_{21} \end{bmatrix}$$

and, again to first order,

$$[OR_T] = [I + \delta_{11} + \delta_{11}^T] \equiv [OR_{11}]$$

Thus, any corrections to the measured modes based only on $[OR_T]$ cannot correct for corruptions due to neglected modes.

We have assumed that the δ_{ij} becomes smaller with increased separation of modal frequencies; however, $(\lambda_{ii} \pm \lambda_{jj})$ becomes large with wide separation, and it is not clear that first-order effects are not being neglected in the resultant E. In Appendix A, it is shown, from a simplified viewpoint at least, that this fear probably is unwarranted.

VI. Correction of Intermediate-Sized Off-Diagonal Elements

At this point, we turn our attention to the practical case of moderate size δ_{ij}. We already have shown for very small errors that their antisymmetric portion is not detectable. With increasing size, as second-order effects begin to take on some significance, antisymmetric errors will reflect their presence into the orthogonality check. The assumption that any off-diagonal terms are so caused will result in the postulation of large antisymmetric errors. Conversely, the assumption that the off-diagonal terms are entirely caused by symmetric errors will result in the postulation of much smaller symmetric errors. In accordance, then, with our ground rule of selecting minimum required changes to the measured modes, we proceed to our correction of intermediate-sized off-diagonal elements with the *a priori* assumption that they are induced completely by symmetric errors. At the worst, this will introduce only a very small extraneous symmetric error into our final result and will leave uncorrected a (perhaps) moderate antisymmetric error, which, however, should have little impact on the resultant E matrix. We assume, then, symmetric errors only. As shown in Eq. (1), $[OR] = C^T C$. Symmetry of C implies $C^T C = C^2$, and the problem of determining the corruption matrix devolves into finding the proper square root of the orthogonality check matrix. There are many ways of doing this. A very quick, simple, iteration scheme is shown below.

Let

$$[OR] = C^2 = (I + \alpha)^2 = I + S$$

Then

$$2\alpha + \alpha^2 = S, \; \alpha(I + \tfrac{1}{2}\alpha) = \tfrac{1}{2}S$$

and

$$\alpha = \tfrac{1}{2}S(I + \tfrac{1}{2}\alpha)^{-1}$$

We iterate by letting

$$\alpha_{N+1} = \tfrac{1}{2}S(I + \tfrac{1}{2}\alpha_N)^{-1} \tag{3}$$

with $\alpha_0 = 0$, $\alpha_1 = \tfrac{1}{2} S$ as in Eq. (2).‡

‡It can be shown by direct substitution in the recursion formula, Eq. (3), that the matrix under inversion is developable as a polynomial in the matrix $S/4$, that it has factors of the form $(I + RS)$, and that, for all N, all of the factors have $0 \le R \le 1$. For the measured modes to be independent, $[I + S]$ must be positive definite, and, therefore, $[I + RS]$ is positive definite. Thus, $(I + \tfrac{1}{2}\alpha_N)^{-1}$ exists for all N.

Before leaving this section, we should note that, if for some reason the analyst wishes to make other assumptions regarding symmetry and asymmetry, his specification of these conditions permits the determination of a C matrix satisfying Eq. (1). For example, the Schmidt[8,9] procedure is equivalent to the assumption of

$$\alpha_{ij}=\beta_{ij}, \quad i<j$$

$$\alpha_{ij}=-\beta_{ij}, \quad i>j$$

Note that, although the assumption of equality of magnitude of the α_{ij} and β_{ij} is inherently satisfying, the assumption that all of the modal errors occur with the same signum disturbs our concepts of the randomness of the errors in the experimental process.

VII. Applications

The method of Sec. VI has been applied successfully to several artificially generated test cases. Some of these, involving matrices of small order, are presented in Appendix B for purposes of illustration. One application to a real problem has been made, with satisfactory results. This involved the ground modal survey of a large spacecraft having 141 degrees of freedom. Seventeen modes were measured, and the orthogonality check revealed smallish but non-negligible off-diagonal elements. Machine computation of Eq. (3) (preset for 10 iterations) resulted in new modes not obviously much different from those measured but having no off-diagonal elements of magnitude greater than 10^{-14}.

VIII. Conclusions

A number of observations may be made regarding the orthogonality check: 1) Small antisymmetric "couplings" do not appear in the check and, therefore, are not correctable. 2) Small symmetric "couplings" are correctable directly. 3) The antisymmetric errors are probably second order in their effect in any use of the measured modes. 4) Small errors in a truncated set caused by higher mode "couplings," cannot be corrected. 5) Correcting the truncated set for small intraset symmetric "couplings" is complete. 6) Simple, workable methods are available for easily correcting moderate-sized symmetric errors.

Appendix A: Estimate of Ratio of Modal Responses

We now shall compute in very simple fashion the response of two modes at the resonance of one of them and take δ_{rs} as the ratio of the two responses

$$\delta_{rs} \propto \frac{1}{1-(\lambda_{ss}/\lambda_{rr})(1+jg)} \div \frac{1}{-jg}$$

Thus,

$$\delta_{rs} \propto \frac{g\lambda_{rr}}{g\lambda_{ss}+j(\lambda_{rr}-\lambda_{ss})}$$

We are interested in the variation of $\alpha_{rs}(\lambda_{rr}+\lambda_{ss})$ and of $\beta_{rs}(\lambda_{rr}-\lambda_{ss})$. Let us assume that δ is either all one or all the other, and see what takes place at two different frequency ratios.

$\alpha_{rs}(\lambda_{rr}+\lambda_{ss})$	$\beta_{rs}(\lambda_{rr}-\lambda_{ss})$
$\propto \frac{g\lambda_{rr}(\lambda_{rr}+\lambda_{ss})}{g\lambda_{ss}+j(\lambda_{rr}-\lambda_{ss})}$	$\propto \frac{g\lambda_{rr}(\lambda_{rr}-\lambda_{ss})}{g\lambda_{ss}+j(\lambda_{rr}-\lambda_{ss})}$
$\propto 2\lambda_{rr}$ (when $\lambda_{ss}\approx\lambda_{rr}$)	$\propto(\lambda_{rr}-\lambda_{ss})\approx 0$ (when $\lambda_{ss}\approx\lambda_{rr}$)
$\propto +jg\lambda_{rr}$ (when $\lambda_{ss}\gg\lambda_{rr}$)	$\propto -jg\lambda_{rr}$ (when $\lambda_{ss}\gg\lambda_{rr}$)

It would appear that neglect of first-order effects is not induced by the truncation process.

Appendix B: Numerical Examples

Case 1

Given the elastic matrix E (with mass matrix identically equal to I) having modes X and roots λ:

$$E=\begin{bmatrix} 35.89 & 62.52 & 13.02 & 0 \\ 62.52 & 112.0 & 31.26 & 0 \\ 13.02 & 31.26 & 46.69 & 27.0 \\ 0 & 0 & 27.0 & 150.7 \end{bmatrix}$$

$$\lambda=\begin{bmatrix} 164.5 & 0 & 0 & 0 \\ 0 & 149.1 & 0 & 0 \\ 0 & 0 & 31.33 & 0 \\ 0 & 0 & 0 & 0.2867 \end{bmatrix}$$

$$X=\begin{bmatrix} 0.3174 & 0.3409 & -0.2378 & -0.8523 \\ 0.5818 & 0.6088 & -0.1767 & 0.5095 \\ 0.3421 & 0.04066 & 0.9316 & -0.1162 \\ 0.6662 & -0.7152 & -0.2108 & 0.02089 \end{bmatrix}$$

Here

$$X\lambda X^T=E+\mathcal{O}(10^{-4})$$

and

$$X^TX=I+\mathcal{O}(10^{-4})$$

Generate, arbitrarily, a corruption matrix C, having a symmetric part CS and a skew symmetric part CA.

$$C=\begin{bmatrix} 0.9796 & 0.0 & 0.04 & 0 \\ 0.2 & 0.9992 & 0 & 0.02 \\ 0 & 0.04 & 0.9992 & 0.04 \\ 0.02 & 0.0 & 0 & 0.9990 \end{bmatrix}$$

$$CS=\begin{bmatrix} 0 & 0.1 & 0.02 & 0.01 \\ 0.1 & 0 & 0.02 & 0.01 \\ 0.02 & 0.02 & 0 & 0.02 \\ 0.01 & 0.01 & 0.02 & 0 \end{bmatrix}$$

$$CA=\begin{bmatrix} 0 & -0.1 & -0.02 & -0.01 \\ 0.1 & 0 & 0.02 & 0.01 \\ -0.02 & 0.02 & 0 & 0.02 \\ 0.01 & -0.01 & -0.02 & 0 \end{bmatrix}$$

Then

$$X_m=\begin{bmatrix} 0.3613 & 0.3311 & -0.2249 & -0.8542 \\ 0.6997 & 0.6013 & -0.1533 & 0.5141 \\ 0.3411 & 0.07786 & 0.9445 & -0.07802 \\ 0.5133 & 0.723 & -0.184 & -0.001845 \end{bmatrix}$$

$$X_M^TX_m=\begin{bmatrix} 1.0 & 0.1958 & 0.03919 & 0.02352 \\ 0.1958 & 1.0 & 0.03995 & 0.02154 \\ 0.03919 & 0.03995 & 1.0 & 0.03994 \\ 0.02352 & 0.02154 & 0.03994 & 1.0 \end{bmatrix}$$

Solution by iteration results in a calculated corruption matrix

$$C_c = \begin{bmatrix} 0.9949 & 0.09816 & 0.0186 & 0.01111 \\ 0.09816 & 0.9949 & 0.01902 & 0.01006 \\ 0.0186 & 0.01902 & 0.9995 & 0.01978 \\ 0.01111 & 0.01006 & 0.01978 & 0.9997 \end{bmatrix}$$

which can be seen to be nearly equal to $I+CS$.

If one now computes the corrected measured modes

$$X_{mc} = X_m C_c^{-1}$$

$$= \begin{bmatrix} 0.3461 & 0.3115 & -0.2204 & -0.8571 \\ 0.648 & 0.5388 & -0.1857 & 0.5053 \\ 0.3235 & 0.02939 & 0.9404 & -0.1005 \\ 0.5964 & 0.7821 & -0.1804 & 0.002968 \end{bmatrix}$$

We compare elastic matrices E_m and E_{mc} based on X_m and X_{mc} with the true E

$$E_m = X_m \lambda X_m^T \qquad E_{mc} = X_{mc} \lambda X_{mc}^T$$

$$E_m - E = \begin{bmatrix} 3.739 & 9.719 & 4.465 & -3.889 \\ 9.719 & 23.27 & 10.44 & -4.854 \\ 4.465 & 10.44 & 1.302 & -12.04 \\ -3.889 & 4.854 & -12.04 & -28.31 \end{bmatrix}$$

$$E_{mc} - E = \begin{bmatrix} 0.03168 & 0.5687 & 0.2947 & -1.125 \\ 0.5687 & 1.52 & 0.09838 & 1.783 \\ 0.2947 & 0.09838 & -1.639 & -4.007 \\ -1.125 & 1.783 & -4.007 & 0.0878 \end{bmatrix}$$

The small residual errors here now are understood to be due to the uncorrected asymmetric errors.

Case 2

Examine another case with significantly larger errors. If we generate

$$C_s = \begin{bmatrix} 0.8246 & -0.4 & 0.4 \\ -0.4 & 0.866 & 0.3 \\ 0.4 & 0.3 & 0.866 \end{bmatrix}$$

With

$$C_s^T C_s = \begin{bmatrix} 1.0 & -0.5563 & 0.5563 \\ -0.5563 & 1.0 & 0.3596 \\ 0.5563 & 0.3596 & 1.0 \end{bmatrix}$$

Iteration returns C_s exactly. If we take

$$C_a = \begin{bmatrix} 0.8246 & 0.4 & 0.4 \\ -0.4 & 0.866 & 0.3 \\ -0.4 & -0.3 & 0.866 \end{bmatrix}$$

$$C_a^T C_a = \begin{bmatrix} 1.0 & 0.1034 & -0.1366 \\ 0.1034 & 1.0 & 0.16 \\ -0.1366 & 0.16 & 1.0 \end{bmatrix}$$

and iteration returns a false

$$C_{sf} = \begin{bmatrix} 0.996 & 0.05488 & -0.0709 \\ 0.05488 & 0.9951 & 0.08239 \\ -0.0709 & 0.08239 & 0.9941 \end{bmatrix}$$

such that $C_{sf}^T C_{sf} = C_a^T C_a$.

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